Chapter 1
Verificationism Then and Now

Per Martin-Löf

The term verificationism is used in two different ways: the first is in relation to the verification principle of meaning, which we usually and rightly associate with the logical empiricists, although, as we now know, it derives in reality from Wittgenstein, and the second is in relation to the theory of meaning for intuitionistic logic that has been developed, beginning of course with Brouwer, Heyting and Kolmogorov in the twenties and early thirties but in much more detail lately, particularly in connection with intuitionistic type theory. It is therefore very natural to ask how these two forms of verificationism are related to one another: was the verificationism that we had in the thirties a kind of forerunner of what we have now, or was it something entirely different? I would like to discuss this question by considering a very particular problem, which was at the heart of Schlick's interests, namely, the problem whether there might exist undecidable propositions or, if you prefer, unsolvable problems or unanswerable questions: it is merely a matter of wording which of these terms you choose. As I said, it is a problem which was at the heart of Schlick's interests: it is explicitly discussed already in his early, programmatic paper *Die Wende der Philosophie* in the first volume of *Erkenntnis* from 1930, and there is a short later paper, which has precisely *Unanswerable Questions?* as its title, from 1935, and he discussed it on several occasions in between also.

So what is the problem? Well, simply this: is it conceivable that some propositions, or some problems, may be such that they just cannot be decided, or cannot be settled, that is, is it conceivable that a proposition may be such that it can neither be proved nor be disproved, or, what amounts to the same, that it can neither be known to be true nor be known to be false? To be very specific, is it, for instance, conceivable that $x^n + y^n$ is in reality different from z^n for arbitrary natural numbers x, y and z when

P. Martin-Löf (✉)
Department of Mathematics, Stockholm University, Stockholm, Sweden
e-mail: pml@math.su.se

M. van der Schaar (ed.), *Judgement and the Epistemic Foundation of Logic*, Logic, Epistemology, and the Unity of Science 31, DOI 10.1007/978-94-007-5137-8_1,

n is greater than 2, but that we are somehow blocked from knowing it, so that all our attempts at trying to prove this will be in vain, or is that conceptually excluded, which is to say that, if this cannot be known to be so, then it is actually false? Or, to vary the example, is it conceivable that there are as a matter of fact infinitely many twin primes, although we cannot prove it by any means, or is that conceptually excluded? Now it is clear from the outset that this is a question about the proper conceptual connections between the notions in terms of which it is formulated, and these are roughly the notions of

proposition,
truth,
falsity,
knowledge,
possibility.

The last of these, the notion of possibility, enters in the guise of the verb can in the formulation of the question: might there exist propositions which *can* neither be known to be true nor be known to be false? So it is a question about the conceptual connections between these few notions, about half a dozen notions, and hence, a definite answer to this question cannot he given unless we decide upon a sufficiently precise interpretation of these notions. The interpretation that I shall develop in the following is the intuitionistic interpretation, and I want to show that, on this interpretation, the question can be definitely answered in the negative.

Now, before the notions of proposition, truth and falsity, basic as they are, there comes in the order of conceptual priority an even more basic notion, namely, the notion of judgement. Indeed, the three first notions on our list, proposition, truth and falsity, are associated with the three forms of judgement

A is a proposition,
A is true,
A is false,

of which the first is used to say that something is a proposition, the second to hold a proposition true and the third to hold a proposition false. Of course, there are many other forms of judgement, for instance, forms of hypothetical and general judgement, and even more elaborate forms of judgement in type theory, but in this talk I will only need to consider these three. So, to our list of notions that need to be clarified, we have to add the notion of judgement.

What is a judgement? Well, the notion of judgement is an essentially epistemic notion, which means that it is connected with the notion of knowledge, the fourth on our list of notions to be clarified, and I think that the most natural explanation is to say that the meaning of a judgement is fixed by laying down what it is that you must know in order to have the right to make the judgement in question. Or, in another formulation, which is the same in substance, though, a judgement is defined simply by what knowledge it embodies: a judgement is a piece of knowledge, and you have to clarify what knowledge.

Now connected with the notion of judgement is the notion of evidence: just as the notion of proposition is coupled with the notion of truth, the notion of judgement is coupled with the notion of being evident, and they are related in the following way: a judgement is evident if it has been known or demonstrated or justified or warranted. There are many terminological possibilities here, but, although there may be different shades of meaning between these in natural language, it does not matter which of these terms I choose in the logical analysis, because it is solely the structure into which they fit which is important, and the structure is one and the same irrespectively of whether I choose to express myself using one or the other of these terms. After all, with the possible exception of known, they are all metaphorical in nature: just as demonstrated is connected with shown, evident is connected with seen, whereas justified and warranted, which was Dewey's preferred term, both seem to be of legal origin.

So much for the notion of evidence of a judgement, but we also have the notion of truth of a judgement. However, since we also have the more well-known notion of truth of a proposition, it is sometimes wise, and quite common, to try to use a different word together with judgements, and the natural choice then is to use correctness, or objective correctness, in connection with judgements. Now what is the connection between the notions of evidence and truth for judgements? Well, simply this: a judgement is true or correct, by definition, if it *can* be made evident. So true or correct for judgements means evidenceable or knowable or demonstrable or justifiable or warrantable: you may choose whichever formulation you prefer here. This analysis of the notion of truth of a judgement in terms of the notions of evidence and possibility validates the Cartesian criterion of truth, which says that, if a judgement is evident, then it is true, in the classical formulation, *quod clare et distincte percipio verum est*, what I clearly and distinctly perceive is true, true in the sense of correct. Indeed, that principle becomes a consequence of my explanation of the notion of truth of a judgement and an even more basic principle, namely, the principle that the scholastics formulated as *ab esse ad posse valet consequentia (illatio).* And why? Because evident means actually known and true means knowable, that is, possibly known, and hence, by the principle that, if something is actual, then it is possible, the Cartesian criterion follows: it becomes simply an instance of the *ab esse ad posse* principle. This will have to be enough about the notions of judgement, evidence and truth of a judgement, so that I can pass on to the notions of proposition and truth.

What is a proposition? Once this question is posed, you see immediately the connection with the general explanation of the notion of judgement that I have just given, and why that explanation had to come first. As I said, a judgement is defined by laying down what it is that you must know in order to have the right to make it, and to ask: what is a proposition? is precisely to ask what you must know in order to have the right to make a judgement of the form A is a proposition, or, equivalently, what knowledge is embodied in a judgement of this form. And here I am going directly to the intuitionistic explanation of the notion of proposition, although we know that it is a relatively late one. So recall the explanations of the meanings of the logical constants, the connectives and the quantifiers, given by Brouwer, Heyting and Kolmogorov: they all follow the common pattern that, whatever the logical constant may be, an explanation is given of what a proof of a proposition formed by means of that logical constant looks like, that is, what is the form, and, more precisely, canonical or direct form, of a proof of a proposition which has that specific logical constant as its outermost sign. It is clear from this what ought to be the general explanation of what a proposition is, namely, that a proposition is defined by stipulating how its proofs, more precisely, canonical or direct proofs, are formed. And, if we take the rules by means of which the canonical proofs are formed to be the introduction rules, I mean, if we call those rules introduction rules as Gentzen did, then his suggestion that the logical constants are defined by their introduction rules is entirely correct, so we may rightly say that a proposition is defined by its introduction rules.

Now what I would like to point out is that this is an explanation which could just as well be identified with the verification principle, provided that it is suitably interpreted. Remember first of all what the verification principle says, namely, that the meaning of a proposition is the method of its verification. The trouble with that principle, considered as a formula, or as a slogan, is that it admits of several different interpretations, so that there arises the question: how is it to be interpreted? Actually, there are at least three natural interpretations of it. On the first of these, the means of verifying a proposition are simply identified with the introduction rules for it, and there is then nothing objectionable about Wittgenstein's formula, provided that we either, as I just did, replace method by means, which is already plural in form, or else make a change in it from the singular to the plural number: the meaning of a proposition is the method*s* of its verification. Interpreted in this way, it simply coincides with the intuitionistic explanation of what a proposition is, or, if you prefer, the Gentzen version of it in terms of introduction rules. For instance, using this manner of speaking, there are two methods of verifying a disjunctive proposition, namely, the two rules of disjunction introduction, and absurdity is defined by stipulating that it admits of no method of verification.

A second interpretation of the term method of verification, perhaps the most natural one, is to use it as a synonym for proof of a proposition, because what is a proof of a proposition on the intuitionistic conception? Well, in general, it need not be in canonical form, that is, it need not have one of the forms displayed in the meaning explanation of the proposition in question, but a proof in general is at least a method which, when it is executed, yields a canonical proof of the proposition as result, so it is very natural to call a proof a method of verification, more precisely, a method of direct, or canonical, verification. But, of course, we are then using the term "method of verification" in a sense which is entirely different from the first one, and which is in conflict with the verification principle.

Now, as a matter of fact, it is in neither of these two senses that the term method of verification was used by Schlick and the Vienna Circle: rather, for them, method of verification meant method of empirical veri- or falsification, that is, method of testing by observation whether the proposition is true or false. So a method of verification was for them simply a decision method, where in addition it is required that the decision, or testing, is to be on empirical grounds. However, in the case of pure mathematics, it is excluded that it could be an empirical testing, so, if we remove that empiricist element, which was absent, by the way, from Wittgenstein's own discussions of the verification principle, what remains is the idea that a method of verification is a method of veri- or falsifying the proposition, that is, a method of deciding whether it is true or false, and such a method is for the intuitionist the same as a proof of $A \vee \neg A$, where A is the proposition in question. Indeed, a proof of $A \vee \neg A$ is a method which, when executed, yields a canonical proof of $A \vee \neg A$ as result, and, by the definition of disjunction, a canonical proof of $A \vee \neg A$ consists either of a proof of A, together with the information that it is a proof of the left disjunct, or of a proof of $\neg A$, together with the information that it is a proof of the right disjunct, so that we can read off which of the two alternatives is the case. To sum up, the outcome of this discussion of the verification principle is that, on the first of the three interpretations that we have considered, the verification principle of meaning is fine as a formulation of what a proposition is on the intuitionistic conception, but that is *not* the interpretation that was actually given to it by the logical empiricists.

Correlated with the verification principle of meaning is the verification principle of truth, which explains what it means for a proposition in the sense that has just been made precise to be true, and the explanation is now very simple, namely, that A is true is taken to mean that there exists a proof of A, a proof which need not necessarily be direct or canonical. The term proof is of course synonymous with verification here. This definition of the notion of truth of a proposition reduces it to two notions, namely, the notion of proof or verification and the notion of existence, and it is because of this that it is very natural to use the term verificationism in connection with the theory of meaning for intuitionistic logic: the term verification is used to stress the fact that the notion of truth is not taken as a primitive notion, like in a truth conditional theory of meaning, but is rather defined in terms of an underlying notion of verification by the principle that A is true if there exists a verification of A. Now, if A is a proposition, then we know of course what a proof of A is, because a proposition is defined precisely by stipulating how its proofs are formed, so we cannot know a proposition without knowing what a proof of the proposition is, but there remains the question how the notion of existence here is to be understood. Normally, we take the notion of existence to be expressed by means of the existential quantifier, and we have a careful explanation of what the existential quantifier means, but it is very clear that the notion of existence as it enters here cannot possibly be expressed by means of the existential quantifier, so we have to give a direct explanation of what we mean by existence here. According to the general explanation of what a judgement is, this means that we have to lay down what it is that you must know in order to have the right to judge that A is true, that is, that there exists a proof of A, and the intuitionist explanation is that to know that there exists a proof of A is to have constructed, or found, a proof of A, that is, to have a proof of A in your possession.

Let me now pass on to the notion of falsity. It has an explanation which is entirely analogous to that of the notion of truth: a proposition A is false, by definition, if there exists a disproof, or refutation, of A. Now I need not say anything more about the notion of existence here, because I have already done that in my discussion of the notion of truth, but, instead, it remains to explain the notion of disproof, or refutation, which is a new notion. And here the explanation is the following: a disproof of a proposition A is a hypothetical proof of absurdity from A. This definition of the notion of disproof presupposes that, among our propositions, we have the special proposition, called absurdity, which is by definition false. Like any other proposition, its meaning is fixed by giving the introduction rules for it, and in this case there are no introduction rules: in the case of disjunction, we have two introduction rules, but, in the case of absurdity, we have zero introduction rules, so the meaning of absurdity is fixed by stipulating that it has no canonical proof and therefore no proof at all. Now, once we have introduced absurdity, symbolized by $\perp$, we can explain the notion of disproof by saying that a disproof of a proposition A is a hypothetical proof of $\perp$ from A, or, what amounts to the same, a function which takes a proof of A into a proof of $\perp$. So, in type theoretical notation, a disproof f of a proposition A is an object of the function type $(A)\perp$:

$$f : (A)\perp .$$

Of course, this constructive notion of falsity, defined in terms of the notion of disproof, or refutation, goes back to Brouwer: to know that a proposition A is false is to have constructed, or found, a refutation of A, that is, to have a refutation of A in your possession.

Once the notion of falsity has been constructed, there arises the question as to what the formal laws are that govern its use. Actually, there are three such laws, and, formulated in natural deduction style, they read as follows. First of all, in addition to the usual assumption rule, which allows us to assume a given proposition to be true, there is a new assumption rule which allows us to assume a given proposition to be false instead. Second, if we have proved, from the assumption that a proposition A is true, that $\perp$ is true, we may conclude that A is false,

$$\frac{\begin{array}{c}(A \text{ true})\\ \perp \text{ true}\end{array}}{A \text{ false}},$$

and, third, if one and the same proposition A has been demonstrated to be both true and false, we may conclude that $\perp$ is true,

$$\frac{A \text{ true} \qquad A \text{ false}}{\perp \text{ true}}.$$

So these are the three formal laws of falsity, provided now that you introduce the notion of falsity into your object language, which is not common, of course: normally, we express the falsity of A by the truth of $\neg A$. Then the rule of assuming a proposition to be false becomes a special case of the usual rule of assuming a proposition to be true, and the two remaining laws of falsity reduce to the negation laws. Now, from these new rules of falsity, it follows immediately that a proposition A is false if and only if $\neg A$ is true, which is to say that the two rules

$$\frac{A \text{ false}}{\neg A \text{ true}}, \qquad \frac{\neg A \text{ true}}{A \text{ false}}$$

are valid as derived rules, and this is of course why it works to define the falsity of A as the truth of $\neg A$, but, nevertheless, falsity is a notion in its own right, and deserves to be treated as such, even if you can do without it from a purely formal point of view.

Now the notion of knowledge, the fourth on our list of notions to be elucidated, I have already dealt with in connection with the notion of judgement: my discussion of the notion of judgement and the notions of evidence and truth, or correctness, of a judgement was a treatment in brief of the epistemic notions that are needed, whereas the notions of proposition, truth and falsity are non-epistemic in nature. And now there remains on our list only the notion of possibility, which I have already used in defining the notion of truth of a judgement as knowability. Concerning this notion of possibility, I have nothing more to say, except that it is the notion of logical possibility, or possibility in principle, as opposed to real, or practical, possibility, which takes resources and so on into account. It is something that was repeated over and over again by Schlick that, in the verification principle, it is absolutely necessary to understand the –able in verifiable as logically possible, or possible in principle, to verify, and, although I am not adhering to the verification principle as interpreted by Schlick, I am as dependent as he was on the notion of logical possibility, or possibility in principle, so I will allow myself to use it without further ado in this discussion.

Now the ordinary logical laws, the laws of propositional and predicate logic, are properly characterized as laws of truth, laws that allow us to derive consequences, which say that one proposition, the consequent, is true provided certain other propositions, the antecedents, are true. It is therefore very natural to ask, once we have seen the correspondence between the non-epistemic notions and the epistemic ones, in particular, between the notion of truth of a proposition and the notion of truth of a judgement, whether there are some general laws that we can formulate for judgements and their truth, which means knowability as we have seen, and indeed there are three such laws. If the ordinary, object linguistic logical laws are characterized as laws of truth, it is natural to refer to these as metalinguistic laws, or laws of knowability. Now the first of these laws is so trivial that maybe it should not be spelled out as a separate law, but I will do it anyway.

First Law (reflection). *If the premises of a valid inference are knowable, then so is the conclusion.*

The justification is simple: if the premises of a valid inference are knowable, or demonstrable, then it is clearly possible to demonstrate, that is, to get to know, the conclusion by first demonstrating the premises and then applying the very inference that is under consideration, the one that is valid by assumption.

The first law, if we choose to call it a law, allows us to lift every object linguistic rule of inference into a metalinguistic rule of inference. So, instead of saying, in an object linguistic mode: $J_1, \ldots, J_n$, therefore J, we say, in a metalinguistic mode: if $J_1, \ldots,$

J_n are knowable, then J is knowable. For example, the usual rule of conjunction introduction

$$\frac{A \text{ true} \qquad B \text{ true}}{A \,\&\, B \text{ true}},$$

is lifted into the metalinguistic law which says: if two propositions A and B can both be known to be true, then $A\&B$ can be known to be true.

Second Law (absolute consistency). *Absurdity cannot be known to be true.*

In other words, the judgement

$$\perp \text{ true}$$

is unknowable. And how do you see this? Well, as always in the case of an axiom, by reflection on the meanings of the terms involved. Remember how absurdity was defined: like any other proposition, it was defined by its introduction rules, and, in the particular case of absurdity, there are none. This means that there is no canonical proof of absurdity, and, since an arbitrary, possibly noncanonical proof is a method, or program, which yields a canonical proof as result, there is no noncanonical proof either. Hence, it is impossible to know a proof of absurdity, and, by the definition of truth, this amounts to the same as saying that it is impossible to know that absurdity is true.

It is noteworthy that the absolute consistency is more basic even than the law of contradiction, in the sense that the law of contradiction follows as a corollary from it.

Corollary (law of contradiction). *One and the same proposition cannot both be known to be true and be known to be false.*

Put differently, the two judgements A true and A false, which presuppose A to be a proposition, are not both knowable, or correct. To see why, remember that

$$\frac{A \text{ true} \qquad A \text{ false}}{\perp \text{ true}}$$

is a valid rule of inference: it is the second of the two rules of inference associated with the notion of falsity. Hence, by the first law of knowability, if the judgements A true and A false are both knowable, then so is the judgement $\perp$ true. But that is excluded by the second law, so A true and A false cannot both be knowable, or correct, which is precisely what the law of contradiction states.

Now, just as the law of contradiction follows as a corollary from the second law, the answer to the question with which I began this talk will follow as a corollary from the third law of knowability.

Third Law (unknowability of truth entails knowability of falsity). *If a proposition cannot be known to be true, then it can be known to be false.*

Since, as we have seen, the judgement A false, where A is a proposition, is interderivable with the judgement $\neg A$ true, the third law may just as well be rendered: if a proposition cannot be known to be true, then its negation can be known to be true. What is more, this is the only formulation available if you choose to define falsity in terms of negation and truth rather than to take it as a primitive notion. And how do you convince yourself of the third law? Well, let a proposition, say A, be given, and suppose that the judgement A true is unknowable. By the definition of truth, knowing that A is true amounts to the same as knowing a proof of A. Hence, using type theoretic notation, the assumption that A cannot be known to be true means that the epistemic situation

$$\mathrm{a} : A$$

cannot arise: it is impossible that we arrive at a judgement of this form. Now, from this negative piece of information, I have to get something positive, namely, I have to show that we actually *can* know a refutation of A, and a refutation of A is a hypothetical proof of $\perp$ from A, or, equivalently, a function which takes a proof of A into a proof of $\perp$. The argument is this: we simply introduce a hypothetical proof of $\perp$ from A, call it R. In type theoretical terms, this means that we introduce an object R of the function type $(A)\,\perp$, in symbols,

$$R : (A)\,\perp,$$

and it only remains for us to make this judgement (in fact, axiom) evident. So what does it mean? Well, by the semantical explanation of the function type, it means that $R(\mathrm{a}) : \perp$ provided that $\mathrm{a} : A$, and, moreover, that $R(\mathrm{a}) = R(\mathrm{b}) : \perp$ provided that $\mathrm{a} = \mathrm{b} : A$. Thus, the crucial judgement $R : (A)\,\perp$ may be considered as a licence to infer by the two rules

$$\frac{\mathrm{a} : A}{R(\mathrm{a}) : \perp}, \qquad \frac{\mathrm{a} = \mathrm{b} : A}{R(\mathrm{a}) = R(\mathrm{b}) : \perp},$$

which are both vacuously valid. This is obvious in the case of the first rule, since, by assumption, its premise can never be demonstrated, and it is equally obvious in the case of the second rule, since its premise carries with it the two presuppositions $\mathrm{a} : A$ and $\mathrm{b} : A$, which can never be demonstrated either. So we may safely judge R to be an object of type $(A)\,\perp$, that is, to be a refutation of the proposition A. This finishes the explanation why a proposition which cannot be known to be true, in recompense, can be known to be false. Observe how similar it is to the justification of the rule of absurdity elimination, the rule that was referred to as *ex falso sequitur quodlibet* by the scholastic logicians.

Corollary (law of excluded middle). *There are no propositions which can neither be known to be true nor be known to be false.*

In short, there are no absolutely undecidable propositions. And why does this follow from the third law? Well, suppose that we had a proposition which could neither be known to be true nor be known to be false. Then, in particular, it cannot be known to be true, so, by the third law, it can instead be known to be false. But that contradicts the assumption that the proposition could not be known to be false either. So the answer to the question with which I began this talk – might there exist absolutely undecidable propositions? – is in the negative, and this is precisely the conclusion, in both senses of the word, that I wanted to reach.

Let me just finish by comparing the preceding treatment with the way in which absolutely undecidable propositions were excluded by Schlick. For him, it was much easier, because a proposition was for him defined by its method of verification, where method of verification was interpreted as method of veri- or falsifying the proposition, that is, as method of deciding whether the proposition is true or false: if it has no clear method of verification, the alleged proposition simply is not a proposition, that is, it is not meaningful. So Schlick's interpretation of the verification principle actually validates the law of excluded middle in its positive formulation, which says that every proposition can either be known to be true or be known to be false, and clearly so: simply execute the method of verification, or decision method, that defines the proposition in question. As concerns the foundations of mathematics, Schlick was most strongly influenced by Hilbert, and at least one source of his interest in the question of unsolvable problems must have been Hilbert's mathematical problems paper from 1900, in which he just states as an axiom, or a conviction, which every mathematician certainly shares, that every mathematical problem can be solved, that is, that every mathematical proposition can either be proved or be disproved. Schlick's way of justifying that axiom was to say that a proposition is defined by its method of verification, that is, by its decision method, and hence, by being a proposition, it is necessarily decidable. Here we see that we have had to go a considerably more roundabout way to reach the weaker conclusion that there are no absolutely undecidable propositions. It is the price that we have had to pay for being able to make sense of quantification over infinite domains, like the domain of the natural numbers. There are many propositions whose meanings we understand perfectly well although we do not known how to decide whether they are true or false.

Postscript, January 2012

As a result of having reread the preceding article after seventeen years, I have become dissatisfied with the treatment of what I called the third law and its corollary, and therefore propose the following amended treatment.

Third Law (unknowability of truth entails falsity). *From the unknowability of the truth of a proposition, its falsity may be inferred.*

Since, as we have seen, the judgement A false is interderivable with the judgement $\neg A$ true, the third law may just as well be rendered: from the unknowability of the truth of a proposition, the truth of its negation may be inferred. What is more, this is the only formulation available if you choose to define falsity in terms of negation and truth rather than to take it as a primitive notion. And how do you convince yourself of the third law? Well, let a proposition, say A, be given, and suppose that the judgement A true is unknowable. By the definition of truth, knowing that A is true amounts to the same as knowing a proof of A. Hence, using type theoretic notation, the assumption that A cannot be known to be true means that the epistemic situation

$$\mathrm{a} : A$$

cannot arise: it is impossible that we arrive at a judgement of this form. Now, from this negative piece of information, I have to get something positive, namely, I have to construct a refutation of A, and a refutation of A is a hypothetical proof of $\perp$ from A, or, equivalently, a function which takes a proof of A into a proof of $\perp$. The argument is this: we simply introduce a hypothetical proof of $\perp$ from A, call it R. In type theoretical terms, this means that we introduce an object R of the function type $(A)\perp$, in symbols,

$$R:(A)\perp,$$

and it only remains for us to make this judgement (in fact, axiom) evident. So what does it mean? Well, by the semantical explanation of the function type, it means that $R(\mathrm{a}):\perp$ provided that $\mathrm{a}:\mathrm{A}$, and, moreover, that $R(\mathrm{a})=R(\mathrm{b}):\perp$ provided that $\mathrm{a}=\mathrm{b}:\mathrm{A}$. Thus, the crucial judgement $R:(A)\perp$ may be considered as a licence to infer by the two rules

$$\frac{\mathrm{a}:\mathrm{A}}{R(\mathrm{a}):\perp}, \qquad \frac{\mathrm{a}=\mathrm{b}:\mathrm{A}}{R(\mathrm{a})=R(\mathrm{b}):\perp},$$

which are both vacuously valid. This is obvious in the case of the first rule, since, by assumption, its premise can never be demonstrated, and it is equally obvious in the case of the second rule, since its premise carries with it the two presuppositions $\mathrm{a}:A$ and $\mathrm{b}:A$, which can never be demonstrated either. So we may safely judge R to be an object of type $(A)\perp$, that is, to be a refutation of the proposition A. It now only remains to make the inference

$$\frac{R:(A)\perp}{A\ \text{false}}$$

in order to reach the desired conclusion that A is false. This finishes the explanation why, from the unknowability of the truth of a proposition, we may infer its falsity. Observe how similar it is to the justification of the rule of absurdity elimination, the rule that was referred to as *ex falso (sequitur) quodlibet* by the scholastic logicians.

Corollary (law of excluded middle). *There are no propositions which can neither be known to be true nor be known to be false.*

In short, there are no absolutely undecidable propositions. Whichever way it is formulated, however, the negative existential: there are no …, as it occurs in either of the two formulations, needs careful explanation, since it cannot be expressed by means of an ordinary negated existential quantifier. What the corollary says, in detail, is that it is impossible to give a counterexample to the law of excluded middle in its positive formulation: every proposition can either be known to be true or be known to be false, which Brouwer correctly identified with Hilbert's solvability axiom. Such a counterexample would have to be a proposition for which it had been established that it can neither be known to be true nor be known to be false. What the corollary says is therefore that the epistemic situation

$$A \text{ prop},$$
$$(A \text{ true}) \text{ unknowable},$$
$$(A \text{ false}) \text{ unknowable},$$

cannot arise. To see why, first apply the third law to the first two judgements in order to reach the conclusion A false, from which $\neg A$ true follows by the first of the two rules

$$\frac{A \text{ false}}{\neg A \text{ true}}, \qquad \frac{\neg A \text{ true}}{A \text{ false}}.$$

Then use the second of these rules to conclude, from the unknowability of A false, that $\neg A$ true is likewise unknowable. A second application of the third law now yields $\neg A$ false. We have thus arrived at both $\neg A$ true and $\neg A$ false, which is impossible by the law of contradiction. The epistemic situation determined by the three judgements above is hence impossible, which is to say that it is impossible to find a counterexample to the law of excluded middle in its positive formulation, and this is precisely what the law of excluded middle in its negative formulation says: tertium non datur.