# Notes on The Domain Interpretation of Type Theory

by

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## 1. Computation rules

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$$\lambda(b) \Rightarrow \lambda(b)$$

$$\frac{c \Rightarrow \lambda(b) \quad d(b) \Rightarrow e}{F(c,d) \Rightarrow e}$$

$$\left( \begin{array}{c} app(c,a) \equiv F(c,(y)y(a)) \\ \dfrac{c \Rightarrow \lambda(b) \quad b(a) \Rightarrow e}{app(c,a) \Rightarrow e} \end{array} \right)$$

$$(a,b) \Rightarrow (a,b)$$

$$\frac{c \Rightarrow (a,b) \quad d(a,b) \Rightarrow e}{E(c,d) \Rightarrow e}$$

$$\left( \begin{array}{c} p(c) \equiv E(c,(x,y)x) \\ q(c) \equiv E(c,(x,y)y) \\ \dfrac{c \Rightarrow (a,b) \quad a \Rightarrow e}{p(c) \Rightarrow e} \qquad \dfrac{c \Rightarrow (a,b) \quad b \Rightarrow e}{q(c) \Rightarrow e} \end{array} \right)$$

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$$i(a) \Rightarrow i(a) \qquad j(b) \Rightarrow j(b)$$

$$(i(a) \equiv inl(a) \qquad j(b) \equiv inr(b))$$

$$\frac{c \Rightarrow i(a) \quad d(a) \Rightarrow f}{D(c,d,e) \Rightarrow f}$$

$$\frac{c \Rightarrow j(b) \quad e(b) \Rightarrow f}{D(c,d,e) \Rightarrow f}$$

$$m_n \Rightarrow m_n \quad (m=0,1,\ldots,n-1)$$

$$\frac{c \Rightarrow m_n \quad c_m \Rightarrow d}{R_n(c,c_0,\ldots,c_{n-1}) \Rightarrow d}$$

$$\left( \begin{array}{l} true \equiv 0_2 \qquad false \equiv 1_2 \\ \text{if } c \text{ then } c_0 \text{ else } c_1 \equiv R_2(c,c_0,c_1) \end{array} \right)$$

$$0 \Rightarrow 0$$

$$s(a) \Rightarrow s(a)$$

$$\frac{c \Rightarrow 0 \quad d \Rightarrow f}{R(c,d,e) \Rightarrow f}$$

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$$\frac{c \Rightarrow s(a) \quad e(a,R(a,d,e)) \Rightarrow f}{R(c,d,e) \Rightarrow f}$$

$$sup(a,b) \Rightarrow sup(a,b)$$

$$\frac{c \Rightarrow sup(a,b) \quad d(a,b,(x)T(b(x),d)) \Rightarrow e}{T(c,d) \Rightarrow e}$$

$$s(a) \Rightarrow s(a)$$

$$\frac{c \Rightarrow s(a) \quad d(a, omegarec(a,d)) \Rightarrow e}{omegarec(c,d) \Rightarrow e}$$

$$\omega \Rightarrow s(\omega)$$

$$\left( \begin{array}{c} fix(f) \equiv omegarec(\omega,(x,y)f(y)) \\ \dfrac{f(fix(f)) \Rightarrow e}{fix(f) \Rightarrow e} \end{array} \right)$$

## 2. (Formal) neighbourhoods

If $U_1, \ldots, U_n$ are subbasic neighbourhoods, then

$$\bigcap_{i<n}^{n} U_i$$

is a neighbourhood. For $n=0$ this clause yields the trivial neighbourhood $\Delta$ (no information).

If $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ are neighbourhoods, then

$$\lambda\left(\bigcap_{i<n}^{n} [U_i, V_i]\right)$$

is a subbasic neighbourhood.

If $U$ and $V$ are neighbourhoods, then

$$(U, V)$$

is a subbasic neighbourhood.





If $U$ is a neighbourhood, then

$$i(U)$$

is a subbasic neighbourhood. Similarly, if $V$ is a neighbourhood, then

$$j(V)$$

is a subbasic neighbourhood.

For $m = 0, \ldots, n-1$,

$$m_n$$

is a subbasic neighbourhood.

$0$ is a subbasic neighbourhood, and if $U$ is a neighbourhood, then

$$s(U)$$

is a subbasic neighbourhood



If $U$ and $V_1, W_1, \ldots, V_n, W_n$ are neighbourhoods, then

$$\sup\left(U, \bigcap_{i<n}^{n} [V_i, W_i]\right)$$

is a subbasic neighbourhood.

Definition of the relation

$$\mathrm{Approx}(a, U) \equiv a \in U$$

which says that a formal neighbourhood $U$ approximates a program $a$.

$$\frac{\&_{i<n}^{n} (a \in U_i)}{a \in \bigcap_{i<n}^{n} U_i} \qquad \text{spec.} \quad a \in \Delta$$

$$\frac{c \Rightarrow \lambda(b) \quad \begin{matrix}(x \in U_1) \\ b(x) \in V_1\end{matrix} \quad \cdots \quad \begin{matrix}(x \in U_n) \\ b(x) \in V_n\end{matrix}}{c \in \lambda\left(\bigcap_{i<n}^{n} [U_i, V_i]\right)}$$



$$\frac{c \Rightarrow (a, b) \quad a \in U \quad b \in V}{c \in (U, V)}$$

$$\frac{c \Rightarrow i(a) \quad a \in U}{c \in i(U)}$$

$$\frac{c \Rightarrow j(b) \quad b \in V}{c \in j(V)}$$

$$\frac{c \Rightarrow m_n}{c \in m_n} \quad (m = 0, \ldots, n-1)$$

$$\frac{c \Rightarrow 0}{c \in 0} \qquad \frac{c \Rightarrow s(a) \quad a \in U}{c \in U} \quad s(U)?$$

$$\frac{c \Rightarrow \sup(a, b) \quad a \in U \quad \begin{matrix}(x \in V_i) \\ b(x) \in W_i\end{matrix} \ (i<n)}{c \in \sup\left(U, \bigcap_{i<n} [V_i, W_i]\right)}$$

If inclusion between and consistency of neighbourhoods are defined by

$$U \subseteq V \equiv (\forall x)(x \in U \supset x \in V),$$

$$\mathrm{Con}\,(U) \equiv (\exists x)(x \in U),$$

then it is clear that the following properties hold,

$$U \subseteq U,$$

$$U \subseteq V \;\&\; V \subseteq W \supset U \subseteq W,$$

$$\bigcap_{i \in I} U_i \subseteq U_i \quad \text{for } i \in I,$$

$$(\forall i \in I)(V \subseteq U_i) \supset (V \subseteq \bigcap_{i \in I} U_i)$$

(in particular, $V \subseteq \Delta$),

$$\mathrm{Con}\,(U) \;\&\; U \subseteq V \supset \mathrm{Con}\,(V)$$

$$\mathrm{Con}\,(\Delta).$$

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With an open program $b(x_1, \ldots, x_n)$, I associate the $(n+1)$-ary relation between neighbourhoods $\tilde{b}$ defined by putting

$$U_1, \ldots, U_n \,\tilde{b}\, V \equiv (\forall x_1 \in U_1) \ldots (\forall x_n \in U_n)(b(x_1, \ldots, x_n) \in V).$$

So defined, it is obviously the case that

$$U_1 \in \tilde{a}_1 \;\&\; \ldots \;\&\; U_n \in \tilde{a}_n \;\& \;\& \; U_1, \ldots, U_n \,\tilde{b}\, V \supset V \in \widetilde{b(a_1, \ldots, a_n)}$$



Moreover, with every program $a$, there is associated the operational neighbourhood filter 26

$$\tilde{a} = \{U \mid a \in U\},$$

which has the defining properties of a filter,

$$U \in \tilde{a} \;\&\; U \subseteq V \supset V \in \tilde{a}$$

$$U_i \in \tilde{a} \text{ for } i \in I \supset \bigcap_{i \in I} U_i \in \tilde{a}$$

(in particular, $\Delta \in \tilde{a}$),

$$U \in \tilde{a} \supset \mathrm{Con}(U).$$

The neighbourhoods define a topology (in the standard sense of set theoretical topology) on the universe of all programs.

## 10 3. Formal inclusion and consistency

$$\mathrm{Con}(\Delta)$$

$$\frac{\mathrm{Con}(\bigcap_{i \in I} W_i)}{\mathrm{Con}(\bigcap_{i \in I} \lambda(W_i))} \qquad W_i = \bigcap_{j \in J_i} [U_{ij}, V_{ij}]$$

$$\frac{(\forall J \subseteq I)(\mathrm{Con}(\bigcap_{i \in J} U_i) \supset \mathrm{Con}(\bigcap_{i \in J} V_i))}{\mathrm{Con}(\bigcap_{i \in I} [U_i, V_i])}$$

$$\frac{\mathrm{Con}(\bigcap_{i \in I} U_i) \quad \mathrm{Con}(\bigcap_{i \in I} V_i)}{\mathrm{Con}(\bigcap_{i \in I} (U_i, V_i))}$$

$$\frac{\mathrm{Con}(\bigcap_{i \in I} U_i)}{\mathrm{Con}(\bigcap_{i \in I} i(U_i))} \qquad \frac{\mathrm{Con}(\bigcap_{i \in I} V_i)}{\mathrm{Con}(\bigcap_{i \in I} j(V_i))} \quad 27$$

$\mathrm{Con}(m_n) \quad (n=0,\dots,n-1)$

$\mathrm{Con}(0)$

$$\frac{\mathrm{Con}(\bigcap_{i\in I} U_i)}{\mathrm{Con}(\bigcap_{i\in I} s(U_i))}$$

$$\frac{\mathrm{Con}(\bigcap_{i\in I} U_i) \quad \mathrm{Con}(\bigcap_{i\in I}\bigcap_{j\in J_i}[V_{ij},W_{ij}])}{\mathrm{Con}(\bigcap_{i\in I} \sup(U_i, \bigcap_{j\in J_i}[V_{ij},W_{ij}]))}$$

For consistency as defined before, $\mathrm{Con}(U) \equiv (\exists x)(x\in U)$, the same laws hold with exception for the case of function neighbourhoods. We <u>do</u> have

$$\mathrm{Con}(\bigcap_{i\in I}[U_i,V_i])$$
$$\supset (\forall J\subseteq I)(\mathrm{Con}(\bigcap_{i\in J} U_i) \supset \mathrm{Con}(\bigcap_{i\in J} V_i))$$

12 but the converse implication need not hold. 13

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Also for function neighbourhoods:

$$\frac{(\forall i\in I)(U\subseteq V_i)}{U\subseteq \bigcap_{i\in I} V_i} \quad \left(\begin{array}{c}\text{in particular,}\\ U\subseteq\Delta\end{array}\right)$$

$$\frac{\bigcap_{i\in J} U_i \subseteq V \quad J\subseteq I}{\bigcap_{i\in I} U_i\subseteq V}$$

$$\frac{\bigcap_{i\in I} W_i\subseteq W}{\bigcap_{i\in I}\lambda(W_i)\subseteq\lambda(W)}$$

$$\frac{U\subseteq\bigcap_{i\in I} U_i \quad \bigcap_{i\in I} V_i\subseteq V}{\bigcap[U_i,V_i]\subseteq[U,V]}$$

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$$\frac{\bigcap_{i\in I} U_i\subseteq U \quad \bigcap_{i\in I} V_i\subseteq V}{\bigcap_{i\in I}(U_i,V_i)\subseteq(U,V)}$$

$$\frac{\bigcap_{i\in I} U_i\subseteq U}{\bigcap_{i\in I} i(U_i)\subseteq U} \qquad \frac{\bigcap_{i\in I} V_i\subseteq V}{\bigcap_{i\in I} j(V_i)\subseteq j(V)}$$

$m_n\subseteq m_n \quad (n=0,\dots,n-1)$

$0\subseteq 0$

$$\frac{\bigcap_{i\in I} U_i\subseteq U}{\bigcap_{i\in I} s(U_i)\subseteq s(U)}$$

function neighbourhood

$$\frac{\bigcap_{i\in I} U_i\subseteq U \quad \bigcap_{i\in I} V_i\subseteq V}{\bigcap_{i\in I}\sup(U_i,V_i)\subseteq\sup(U,V)}$$

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**Theorem**

If $U$ is formally included in $V$, then

$$(\forall x)(x\in U\supset x\in V).$$

<u>Proof.</u> Consider the case of function neighbourhoods.

Assume

$$(\forall x)(x\in U\supset x\in\bigcap_{i\in I} U_i)$$

$$(\forall y)(y\in\bigcap_{i\in I} V_i\supset y\in V)$$

$$(\forall i\in I)(\forall x)(x\in U_i\supset f(x)\in V_i)$$

Want to prove

$$(\forall x)(x\in U\supset f(x)\in V).$$

Clear!

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**Theorem.** If $U$ is formally consistent and formally included in $V$, then $V$ is formally consistent.

Proof.

$$U = \Delta \subseteq V \to V = \Delta$$

$\bigcap_{i\in I} U_i$ with $I$ empty

$$U = \bigcap_{i\in I\neq\emptyset} \lambda(U_i) \subseteq V = \bigcap_{j\in J} V_j$$

$J = \emptyset \to V = \Delta$ Trivial!

$J \neq \emptyset$

$\therefore V_j = \lambda(W_j)$

Want to prove that $\mathrm{Cons}(\bigcap_{j\in J} W_j)$ follows from $\mathrm{Cons}(\bigcap_{i\in I} U_i)$.

17 [30]

$$\mathrm{Cons}\Big(\bigcap_{i\in I}[U_i, V_i]\Big)$$
$$\&\Big(\bigcap_{i\in I}[U_i, V_i] \subseteq \bigcap_{j\in J}[W_j, Z_j]\Big)$$
$$\supset \mathrm{Cons}\Big(\bigcap_{j\in J}[W_j, Z_j]\Big)$$

$$W_j \subseteq \bigcap_{i\in I_j} U_i \ \&\ \bigcap_{i\in I_j} V_i \subseteq Z_j \quad (j\in J)$$

$$\mathrm{Cons}\Big(\bigcap_{j\in K\subseteq J} W_j\Big)$$

$$\bigcap_{j\in K} W_j \subseteq \bigcap_{j\in K}\ \bigcap_{i\in I_j} U_i$$

$$\therefore \mathrm{Cons}\Big(\bigcap_{j\in K}\ \bigcap_{i\in I_j} U_i\Big)$$

$$\therefore \mathrm{Cons}\Big(\bigcap_{j\in K}\ \bigcap_{i\in I_j} V_i\Big)$$

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$\mathrm{Cons}(U)$

$$U = \bigcap_{i\in I\neq\emptyset} s(U_i) \subseteq V = \bigcap_{j\in J} V_j$$

$$V_j = s(W_j)$$

$$\bigcap_{i\in I} U_i \subseteq W_j \quad (j\in J)$$

$$\bigcap_{i\in I} U_i \subseteq \bigcap_{j\in J} W_j$$

$$\therefore \mathrm{Cons}\Big(\bigcap_{j\in J} W_j\Big)$$

$$\therefore \underbrace{\mathrm{Cons}\Big(\bigcap_{j\in J} \underbrace{s(W_j)}_{=V_j}\Big)}_{=V}$$

etc.

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Thus the relation of formal inclusion between and the property of formal consistency of neighbourhoods satisfy the laws established earlier for $U \subseteq V \equiv (\forall x)(x\in U \supset x\in V)$ and $\mathrm{Con}(U) \equiv (\exists x)(x\in U)$, that is,

$U \subseteq U$

$U \subseteq V \ \&\ V \subseteq W \supset U \subseteq W$,

$(\forall i\in I)(\bigcap_{i\in I} U_i \subseteq U_i)$,

$(\forall i\in I)(V \subseteq U_i) \supset (V \subseteq \bigcap_{i\in I} U_i)$

(in particular, $V \subseteq \Delta$),

$\mathrm{Con}(\Delta)$,

$\mathrm{Con}(U) \ \&\ U \subseteq V \supset \mathrm{Con}(V)$. [31]

Moreover, formal inclusion and consistency are decidable.

A <u>neighbourhood filter</u> is a set $\alpha$ of neighbourhoods satisfying the three conditions:

formal inclusion

(1) $U \in \alpha \ \& \ U \subseteq V \supset V \in \alpha$.

(2) If $U_i \in \alpha$ for $i \in I$, then $\bigcap_{i \in I} U_i \in \alpha$. In particular for $I = \emptyset$, we have $\Delta \in \alpha$.

(3) $U \in \alpha \supset \mathrm{Con}(U)$.

formal consistency

By induction on its construction, I associate with a program $a$ its denotation $\hat{a}$, which is a neighbourhood filter.

If the program is of the form, say $b(x_1, \ldots, x_n)$, its denotation



$\hat{b}$ is an $(n+1)$-ary relation between neighbourhoods $U_1, \ldots, U_n \hat{b} V$ satisfying the conditions: $\boxed{32}$

(1) $U_1 \subseteq V_1 \ \& \ldots \& \ U_n \subseteq V_n \ \&$ $\& \ V_1, \ldots, V_n \hat{b} W \supset U_1, \ldots, U_n \hat{b} W$.

(2) $U_1, \ldots, U_n \hat{b} V \ \& \ V \subseteq W$ $\supset U_1, \ldots, U_n \hat{b} W$.

(3) $U_1, \ldots, U_n \hat{b} V_j$ for $j \in J$ $\supset U_1, \ldots, U_n \hat{b} \bigcap_{j \in J} V_j$ ($U_1, \ldots, U_n \hat{b} \Delta$ in particular).

(4) $\mathrm{Con}(U_1) \ \& \ldots \& \ \mathrm{Con}(U_n)$ $\& \ U_1, \ldots, U_n \hat{b} V \supset \mathrm{Con}(V)$.

Moreover the denotation of a program is defined in such a way that it is obvious that



$$(\exists U_1 \in \hat{a}_1) \ldots (\exists U_n \in \hat{a}_n)(U_1, \ldots, U_n \hat{b} V) \supset\subset V \in \widehat{b(a_1, \ldots, a_n)}.$$

Compare this with the one way implication

$$(\exists U_1 \in \tilde{a}_1) \ldots (\exists U_n \in \tilde{a}_n)(U_1, \ldots, U_n \tilde{b} V) \supset V \in \widetilde{b(a_1, \ldots, a_n)}$$

established earlier. The converse implication would say that $b(x_1, \ldots, x_n)$ is <u>continuous</u> in the standard sense of set theoretical topology. We avoid having to prove this by working with $\hat{b}$ instead of $\tilde{b}$.



$$\widehat{\lambda(b)} = \{\lambda(\bigcap_{i \in I} [U_i, V_i]) \mid (\forall i \in I)\hat{b}(U_i, V_i)\} \cup \{\Delta\}$$

$$\widehat{app(c,a)} = \{V \mid (\exists U \in \hat{a})(\lambda([U,V]) \in \hat{c})\} \cup \{\Delta\}$$

$$\widehat{(a,b)} = \{(U,V) \mid U \in \hat{a} \ \& \ V \in \hat{b}\} \cup \{\Delta\}$$

$$\widehat{E(c,d)} = \{W \mid (\exists U,V)((U,V) \in \hat{c} \ \& \ \hat{d}(U,V,W))\} \cup \{\Delta\}$$

$$\widehat{i(a)} = \{i(U) \mid U \in \hat{a}\} \cup \{\Delta\}$$

$$\widehat{j(b)} = \{j(V) \mid V \in \hat{b}\} \cup \{\Delta\}$$

$$\widehat{D(c,d,e)} = \{W \mid (\exists U)(i(U) \in \hat{c} \ \& \ \hat{d}(U,W)) \vee (\exists V)(j(V) \in \hat{c} \ \& \ \hat{e}(V,W))\} \cup \{\Delta\}$$

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$\hat{m}_n = \{\Delta, m_n\}$ $(m=0,\dots,n-1)$

$\widehat{R_n(c, c_0, \dots, c_{n-1})} = \{W \mid (0_n \in \hat{c} \,\&\, W \in \hat{c}_0) \vee \dots \vee ((n-1)_n \in \hat{c} \,\&\, W \in \hat{c}_{n-1})\} \cup \{\Delta\}$

$\hat{0} = \{\Delta, 0\}$

$\widehat{s(a)} = \{s(u) \mid u \in \hat{a}\} \cup \{\Delta\}$

$\widehat{R(c,d,e)} = \{W \mid (\exists u)(u \in \hat{c} \,\&\, W \in \hat{R}(u, \hat{d}, \hat{e}))\}$

$\hat{R}(\Delta, \hat{d}, \hat{e}) = \{\Delta\}$

$\hat{R}(0, \hat{d}, \hat{e}) = \hat{d}$

$\hat{R}(s(u), \hat{d}, \hat{e}) = \hat{e}(u, \hat{R}(u, \hat{d}, \hat{e}))$

$\hat{R}(u, \hat{d}, \hat{e}) = \{\Delta\}$

consistent neighbourhood which does not have zero or successor form



$\widehat{\mathrm{omegarec}(c,d)} = \{W \mid (\exists u)(u \in \hat{c} \,\&\, W \in \widehat{\mathrm{omegarec}}(u, \hat{d}))\}$



$\widehat{\mathrm{omegarec}}(\Delta, \hat{d}) = \{\Delta\}$

$\widehat{\mathrm{omegarec}}(s(u), \hat{d}) = \hat{d}(u, \widehat{\mathrm{omegarec}}(u, \hat{d}))$

$\widehat{\mathrm{omegarec}}(u, \hat{d}) = \{\Delta\}$

consistent neighbourhood which does not have successor form

$\hat{\omega} = \{\Delta, s(\Delta), s(s(\Delta)), \dots\}$



5. First theorem. If $a \Rightarrow b$, then $\hat{a} = \hat{b}$.

Proof. I have to prove that the denotation $\hat{a}$ remains unchanged under conversion of $a$.

$\widehat{\mathrm{app}(\lambda(b), a)} = \widehat{b(a)}$

$\widehat{b(a)} = \{V \mid (\exists u \in \hat{a})\, \hat{b}(u, V)\}$

$\widehat{\mathrm{app}(\lambda(b), a)} = \{V \mid (\exists u \in \hat{a})\, \underbrace{(\lambda([u,V]) \in \widehat{\lambda(b)})}_{\Leftrightarrow\ \hat{b}(u,V)}\} \cup \{\Delta\}$

$\Delta \in \widehat{b(a)}$ since $\Delta \in \hat{a}$ and $\hat{b}(\Delta, \Delta)$ both hold.

$\widehat{E((a,b),d)} = \widehat{d(a,b)}$

$\widehat{E((a,b),d)} = \{W \mid (\exists u,v)(\underbrace{(u,v) \in \widehat{(a,b)}}_{\Leftrightarrow\ u \in \hat{a} \,\&\, v \in} \,\&\, \hat{d}(u,v,W))\} \cup \{\Delta\}$

$\widehat{D(i(a),d,e)} = \widehat{d(a)}$

$\hookrightarrow \{W \mid (\exists u)(\underbrace{i(u) \in \widehat{i(a)}}_{\Leftrightarrow\ u \in \hat{a}} \,\&\, \hat{d}(u,W)) \vee (\exists v)(\underbrace{j(v) \in \widehat{i(a)}}_{\Leftrightarrow\ \perp} \,\&\, \hat{e}(v,W))\} \cup \{\Delta\}$

$\widehat{D(j(b),d,e)} = \widehat{e(b)}$ is proved in the same way.

$\widehat{R_n(m_n, c_0, \dots, c_{n-1})} = \hat{c}_m$ $(m=0,\dots,n-1)$

$\hookrightarrow \{W \mid (0_n \in \hat{m}_n \,\&\, W \in \hat{c}_0) \vee \dots \vee ((n-1)_n \in \hat{m}_n \,\&\, W \in \hat{c}_{n-1})\} \cup \{\Delta\}$

$\widehat{R(0,d,e)} = \hat{d}$

$\hookrightarrow \{W \mid (\exists u)(u \in \hat{0} \,\&\, W \in \hat{R}(u, \hat{d}, \hat{e}))\}$
$= \{\Delta\} \cup \hat{d} = \hat{d}$

$$\widehat{R(s(a),d,e)} = \widehat{e(a,R(a,d,e))}$$

If $\hat{a}$ is finite, say $u$,

$$\widehat{R(s(u),d,e)} = \hat{R}(s(u),\hat{d},\hat{e})$$
$$= \hat{e}(u,\hat{R}(u,\hat{d},\hat{e})) = \widehat{e(u,R(u,d,e))}$$

holds by virtue of the definition of $\hat{R}(u,\hat{d},\hat{e})$. So, by continuity, it holds for arbitrary $a$.

$$\widehat{\text{omegarec}(s(a),d)}$$
$$= \widehat{d(a,\text{omegarec}(a,d))}$$

is proved in the same way, by omission of the base clause.

$$\hat{\omega} = \widehat{s(\omega)}$$

$$\hat{\omega} = \{\Delta, s(\Delta), s(s(\Delta)), \dots\}$$
$$= \{\Delta\} \cup \{s(\Delta), s(s(\Delta)), \dots\}$$
$$= \{\Delta\} \cup \{s(u) \mid u \in \hat{\omega}\} = \widehat{s(\omega)}$$



6. Second theorem. For an arbitrary program $a$, $\hat{a} \subseteq \tilde{a}$. That is, a denotational neighbourhood of a program is always an operational neighbourhood of the same program.

$$u \in \hat{a} \to u \in \tilde{a} \equiv a \in u$$

Proof. By induction on the construction of the program $a$.

Case 1. $u \in \widehat{\lambda(b)} \to \lambda(b) \in u$.

$u = \Delta$ Trivial!

$u = \lambda(\bigcap_{i \in I}[u_i, v_i])$ where

$(\forall i \in I)\, \hat{b}(u_i, v_i)$

By induction hypothesis, $\tilde{b}(u_i, v_i) \equiv (\forall x \in u_i)(b(x) \in v_i)$ holds for all $i \in I$.



$\therefore\ \lambda(b) \in \lambda(\bigcap_{i \in I}[u_i, v_i]) = u$

Case 2.

$V \in \widehat{app(c,a)} \to app(c,a) \in V$

$V = \Delta$. Trivial!

$(\exists u)(u \in \hat{a}\ \&\ \lambda([u,V]) \in \hat{c})$

By induction hypothesis, $c \in \lambda([u,V])$ and $a \in u$.

$\therefore\ c \Rightarrow \lambda(b)$; $b \in [u,V] \equiv$

$\equiv (\forall x)(x \in u \supset b(x) \in V)$

$\therefore\ b(a) \in V \leftarrow$ proper

$\therefore\ b(a) \Rightarrow d \in V$

$\therefore\ app(c,a) \Rightarrow d \in V$

$\therefore\ app(c,a) \in V$

Case 3. $W \in \widehat{(a,b)} \to (a,b) \in W$

$W = \Delta$ trivial. So assume $W$ proper.

$W = (U,V)$, where $U \in \hat{a}$ & $V \in \hat{b}$

By induction hypothesis, $a \in U$ & $b \in V$.

$\therefore (a,b) \in (U,V) = W$.

Case 4. $W \in \widehat{E(c,d)} \to E(c,d) \in W$

$W = \Delta$ trivial. So assume $W$ proper.

$(\exists U,V)((U,V) \in \hat{c}$ & $\hat{d}(U,V,W))$

By induction hypothesis, $c \in (U,V)$ and $\underbrace{(\forall x \in U)(\forall y \in V)(d(x,y) \in W)}_{\equiv \tilde{d}(U,V,W)}$

$\therefore c \Rightarrow (a,b)$, $a \in U$ & $b \in V$

$\therefore d(a,b) \in W$

$\therefore d(a,b) \Rightarrow e \in W$

$\therefore E(c,d) \Rightarrow e \in W$



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Case 5. $W \in \widehat{i(a)} \to i(a) \in W$

$W \in \widehat{j(b)} \to j(b) \in W$

$W = \Delta$ Trivial!

$W$ proper

$W = i(U)$ where $U \in \hat{a}$

By induction hypothesis, $a \in U$.

$\therefore i(a) \in i(U) = W$

Case 6.

$W \in \widehat{D(c,d,e)} \to D(c,d,e) \in W$

$W = \Delta$ Trivial.

$W$ proper

$(\exists U)(i(U) \in \hat{c}$ & $\hat{d}(U,W)) \vee$
$(\exists V)(j(V) \in \hat{c}$ & $\hat{e}(V,W))$

By induction hypothesis, $c \in i(U)$

$\therefore c \Rightarrow i(a)$, $a \in U$



$\tilde{d}(U,W) \equiv (\forall x)(x \in U \to d(x) \in W)$

$\therefore d(a) \in W$

$\therefore d(a) \Rightarrow f \in W$

$\therefore D(c,d,e) \Rightarrow f \in W$

$\therefore D(c,d,e) \in W$

Case 7. $W \in \hat{m}_n \to m_n \in W$

We even have $\hat{m}_n = \{\Delta, m_n\} = \tilde{m}_n$.

Case 8.

$W \in \widehat{R_n(c,c_0,\dots,c_{n-1})}$

$\to R_n(c,c_0,\dots,c_{n-1}) \in W$

$W = \Delta$ Trivial!

$W$ proper

$(\exists m = 0,\dots,n-1)(m_n \in \hat{c}$ & $W \in \hat{c}_m)$

By induction hypothesis,

$m_n \in \tilde{c} \equiv c \in m_n \quad \therefore c \Rightarrow m_n$

$W \in \tilde{c}_m \equiv c_m \in W \quad \therefore c_m \Rightarrow d \in W$



$\therefore R_n(c,c_0,\dots,c_{n-1}) \Rightarrow d \in W$

$\therefore R_n(c,c_0,\dots,c_{n-1}) \in W$

Case 9. $W \in \hat{0} \to 0 \in W$

We even have $\hat{0} = \{\Delta, 0\} = \tilde{0}$.

Case 10. $W \in \widehat{s(a)} \to s(a) \in W$

The case when $W = \Delta$ is trivial. So assume $W$ proper.

$W = s(U)$, $U \in \hat{a}$

By induction hypothesis, $a \in U$.

$\therefore s(a) \in s(U) = W$

Case 11. $W \in \widehat{R(c,d,e)} \to$

$\to W \in \widehat{R(c,d,e)} \equiv R(c,d,e) \in W$.

Assume $W \in \widehat{R(c,d,e)} \equiv (\exists U)$
$(U \in \hat{c}$ & $W \in \hat{R}(U,\hat{d},\hat{e}))$

By induction hypothesis,
$U \in \tilde{c} \equiv c \in U$

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I prove that

$$c \in U \ \& \ W \in \hat{R}(U, \hat{d}, \hat{e}) \rightarrow R(c,d,e) \in W$$

by induction on $U$. If $W = \Delta$, the conclusion is trivial, so assume ~~The~~ the cases when $U$ is $\Delta$ or does not have zero or successor form do not arise, because then $W = \Delta$. It remains to consider the cases when $U$ has zero or successor form.

$$c \in 0 \quad \therefore c \Rightarrow 0$$

$$W \in \hat{R}(0, \hat{d}, \hat{e}) = \hat{d}$$

By induction hypothesis,

$$d \in W \leftarrow \text{proper}$$

$$\therefore d \Rightarrow f \in W$$

$$\therefore R(c,d,e) \Rightarrow f \in W$$

$$\therefore R(c,d,e) \in W$$





$$c \in s(U) \quad \therefore c \Rightarrow s(a), \ a \in U$$

$$W \in \hat{R}(s(U), \hat{d}, \hat{e}) =$$

$$= \hat{e}(U, \hat{R}(U, \hat{d}, \hat{e}))$$

$$\therefore (\exists V \in \hat{R}(U, \hat{d}, \hat{e}))(W \in \hat{e}(U, V))$$

By the subordinate induction hypothesis on $U$,

$$R(a,d,e) \in V.$$

So, by the principal induction hypothesis,

$$e(a, R(a,d,e)) \in W$$

$$\therefore e(a, R(a,d,e)) \Rightarrow f \in W$$

$$\therefore R(c,d,e) \Rightarrow f \in W$$

$$\therefore R(c,d,e) \in W$$



<u>Case 12</u>. $W \in \overline{\text{omegarec}(c,d)}$

$$\rightarrow W \in \overline{\text{omegarec}(c,d)} \equiv$$

$$\equiv \text{omegarec}(c,d) \in W$$

Like the previous case.

<u>Case 13</u>. $\hat{\omega} \subseteq \tilde{\omega}$

In fact, we even have

$$\hat{\omega} = \{\Delta, s(\Delta), s(s(\Delta)), \cdots\} = \tilde{\omega}.$$

$s^n(\Delta) \in \tilde{\omega} \equiv \omega \in s^n(\Delta)$ is proved by induction on $n$.

$U \in \tilde{\omega} \rightarrow (\exists n)(U = s^n(\Delta))$ is proved by induction on $U$.

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Corollary.

$\lambda(\bigcap_{i\in I}[U_i, V_i]) \in \hat{c} \leftrightarrow (\exists b)(c \Rightarrow \lambda(b)\ \&$
$\&\ (\forall i \in I)\ \hat{b}(U_i, V_i))$

$(U, V) \in \hat{c} \leftrightarrow (\exists a, b)(c \Rightarrow (a, b)\ \&$
$\&\ U \in \hat{a}\ \&\ V \in \hat{b})$

$i(U) \in \hat{c} \leftrightarrow (\exists a)(c \Rightarrow i(a)\ \&\ U \in \hat{a})$
$j(V) \in \hat{c} \leftrightarrow (\exists b)(c \Rightarrow j(b)\ \&\ V \in \hat{b})$

$m_n \in \hat{c} \leftrightarrow c \Rightarrow m_n$

$0 \in \hat{c} \leftrightarrow c \Rightarrow 0$

$s(U) \in \hat{c} \leftrightarrow (\exists a)(c \Rightarrow s(a)\ \&\ a \in U)$

Observe that, when ^ is replaced by ~, these equivalences hold trivially by virtue of the definition of $\tilde{a}$.

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Corollary.

$\hat{c} = \{\Delta\} \leftrightarrow c$ does not termina

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## 3. The Domain Interpretation of Type Theory

Discussion during Per Martin-Löf's talk

(Context: In total type theory, it is possible to show, for example:

$c = \lambda((x)\mathrm{app}(c,x))$

but in partial type theory $c$ may fail to terminate, while $\lambda((x)\mathrm{app}(c,x))$ is on canonical form thus we want to say $c \leq \lambda((x)\mathrm{app}(c,x))$, i.e., $c$ is less defined than $\lambda((x)\mathrm{app}(c,x))$.)

Plotkin: Could you instead change the operational interpretation?

Martin-Löf: Yes - the question is how to get out of this difficulty - either change the notion of computation - or exclude the laws for equality. I cannot give all the arguments here but my choice is to exclude these laws.

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(Context: Contrasting the meaning of the judgements "c is a member of type A" and "c is equal to c′ in type A" in total type theory and partial type theory the following example was discussed from total type theory:

$c \in \Pi(A,B)$

then

$c \Rightarrow \lambda(b)$ ( the value of $c$ is $\lambda(b)$)

where:

$b(x) \in B(x)$ $(x \in A)$

Where this means:

if $a \in A$ then $b(a) \in B(a)$

and

if $a = a' \in A$ then $b(a) = b(a') \in B(a)$

The judgement $c = c' \in A$ was also explained.)

Plotkin: This is an explanation of certain things in terms of prima faciae simpler things, is that the character of it?

Martin-Löf: Yes - The types 'go down' here, it is a predicative theory so the types are well founded. This kind of explanation will not work for impredicative theories.

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